Solved Problems

SAMPLE SPACES AND EVENTS

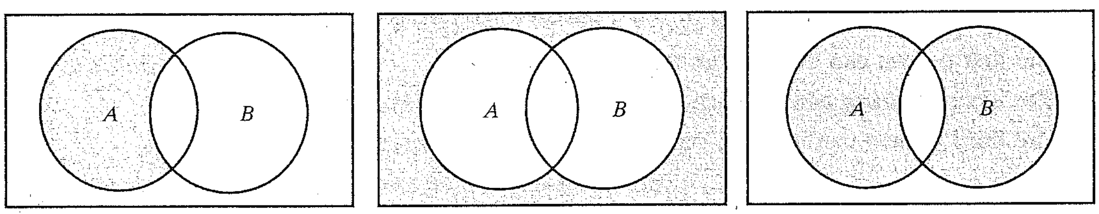
3.1. Let A and B be events. Find an expression and exhibit the Venn diagram for the event:

(a) A but not B, (b) neither A nor B, (c) either A or B, but not both.

(a) Since A but not B occurs, shade the area of A outside of B, as in Fig. 3-7(a). Note that B^c, the complement of B, occurs, since B does not occur; hence A and B^c occur. In other words, the event is $A \cap B^c$.

(b) "Neither A nor B" means "not A and not B" or $A^c \cap B^c$. By DeMorgan's law, this is also the set $(A \cup B)^c$; hence shade the area outside of A and outside of B, that is, outside $A \cup B$, as in Fig. 3-7(b).

(c) Since A or B, but not both, occurs, shade the area of A and B, except where they intersect, as in Fig. 3-7(c). The event is equivalent to the occurrence of A but not B or B but not A. Thus, the event is $(A \cap B^c) \cup (B \cap A^c)$. Alternately, the event is $A \oplus B$, the symmetric difference of A and B.



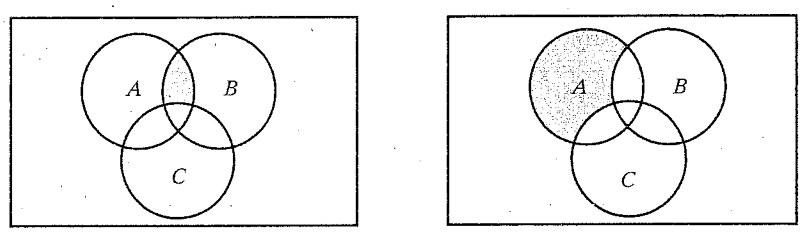
(a) A but not B. (b) Neither A nor B. (c) A or B, but not both.

Fig. 3-7

3.2. Let A, B, C be events. Find an expression and exhibit the Venn diagram for the event:

(a) A and B but not C occurs, (b) only A occurs.

(a) Since A and B but not C occurs, shade the intersection of A and B which lies outside of C, as in Fig. 3-8(a). The event consists of the elements in A, in B, and in C^c (not in C), that is, the event is the intersection $A \cap B \cap C^c$.



(a) A and B but not C occurs. (b) Only A occurs.

Fig. 3-8

(b) Since only A is to occur, shade the area of A which lies outside of B and C, as in Fig. 3-8(b). The event consists of the elements in A, in B^c (not in B), and in C^c (not in C); that is, the event is the intersection $A \cap B^c \cap C^c$.

3.3. Let a coin and a die be tossed; and let the sample space S consist of the 12 elements:

$$S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$$

Express explicitly the following events:

(*a*) A = {heads and an even number}, (*b*) B = {a number less than 3},
(*c*) C = {tails and an odd number}.

(*a*) The elements of A are those elements of S which consist of an H and an even number; hence

$$A = \{H2, H4, H6\}$$

(*b*) The elements of B are those elements of S whose second component is less than 3, that is, 1 or 2; hence

$$B = \{H1, H2, T1, T2\}$$

(*c*) The elements of C are those elements of S which consist of a T and an odd number; hence

$$C = \{T1, T3, T5\}$$

3.4. Consider the events A, B, C in the preceding Problem 3.3. Express explicitly the event that:

(*a*) A or B occurs. (*b*) B and C occur. (*c*) Only B occurs.

Which pair of the events A, B, C are mutually exclusive?

(*a*) "A or B" is the set union $A \cup B$; hence

$$A \cup B = \{H2, H4, H6, H1, T1, T2\}$$

(*b*) "B and C" is the set intersection $B \cap C$; hence

$$B \cap C = \{T1\}$$

(*c*) "Only B" consists of the elements of B which are not in A and not in B, that is, the set intersection $B \cap A^c \cap C^c$; hence

$$B \cap A^c \cap C^c = \{H1, T2\}$$

Only A and C are mutually exclusive, that is, $A \cap C = \varnothing$.

3.5. A pair of dice is tossed and the two numbers appearing on the top are recorded. Recall that S consists of 36 pairs of numbers which are pictured in Fig. 3-3. Find the number of elements in each of the following events:

(*a*) A = {two numbers are equal} (*c*) C = {5 appears on first die}
(*b*) B = {sum is 10 or more} (*d*) D = {5 appears on at least one die}

Use Fig. 3-3 to help count the number of elements in each of the events:

(*a*) $A = \{(1, 1), (2, 2), \ldots, (6, 6)\}$, so $n(A) = 6$.
(*b*) $B = \{(6, 4), (5, 5), (4, 6), (6, 5), (5, 6), (6, 6)\}$, so $n(B) = 6$.
(*c*) $C = \{(5, 1), (5, 2), \ldots, (5, 6)$, so $n(C) = 6$.
(*d*) There are six pairs with 5 as the first element, and six pairs with 5 as the second element. However, $(5, 5)$ appears in both places. Hence

$$n(D) = 6 + 6 - 1 = 11$$

Alternately, count the pairs in Fig. 3-3 which are in D to get $n(D) = 11$.

FINITE EQUIPROBABLE SPACES

3.6. Determine the probability p of each event:

(*a*) An even number appears in the toss of a fair die.

(*b*) At least one tail appears in the toss of 3 fair coins.

(*c*) A white marble appears in the random drawing of 1 marble from a box containing 4 white, 3 red, and 5 blue marbles.

Each sample space S is an equiprobable space. Hence, for each event E, use

$$P(E) = \frac{\text{number of elements in } E}{\text{number of elements in } S} = \frac{n(E)}{n(S)}$$

(*a*) The event can occur in three ways (a 2, 4, or 6) out of 6 equally like cases; hence $p = 3/6 = 1/2$.

(*b*) Assuming the coins are distinguished, there are 8 cases:

$$HHH,\ HHT,\ HTH,\ HTT,\ THH,\ THT,\ TTH,\ TTT$$

Only the first case is not favorable; hence $p = 7/8$.

(*c*) There are $4 + 3 + 5 = 12$ marbles of which 4 are white; hence $p = 4/12 = 1/3$.

3.7. A single card is drawn from an ordinary deck S of 52 cards. (See Fig. 3-4.) Find the probability p that the card is a: (*a*) king, (*b*) face card (jack, queen, or king), (*c*) red card (heart or diamond), (*d*) red face card.

Here $n(S) = 52$.

(*a*) There are 4 kings; hence $p = 4/52 = 1/13$.

(*b*) There are $4(3) = 12$ face cards; hence $p = 12/52 = 3/13$.

(*c*) There are 13 hearts and 13 diamonds; hence $p = 26/52 = 1/2$.

(*d*) There are 6 face cards which are red; hence $p = 6/52 = 3/26$.

3.8. Consider the sample space S and events A, B, C in Problem 3.3 where a coin and a die are tossed. Suppose the coin and die are fair; hence S is an equiprobable space. Find:

(*a*) $P(A)$, $P(B)$, $P(C)$, (*b*) $P(A \cup B)$, $P(B \cap C)$, $P(B \cap A^c \cap C^c)$

Since S is an equiprobable space, use $P(E) = n(E)/n(S)$. Here $n(S) = 12$. We need only count the number of elements in each given set, and then divide by 12.

(*a*) By Problem 3.3, $P(A) = 3/12$, $P(B) = 4/12$, $P(C) = 3/12$.

(*b*) By Problem 3.4, $P(A \cup B) = 6/12$, $P(B \cap C) = 1/12$, $P(B \cap A^c \cap C^c) = 2/12$.

3.9. A box contains 15 billiard balls which are numbered from 1 to 15. A ball is drawn at random and the number recorded. Find the probability p that the number is:

(*a*) even, (*b*) less than 5, (*c*) even and less than 5, (*d*) even or less than 5.

(*a*) There are 7 numbers, 2, 4, 6, 8, 10, 12, 14, which are even; hence $p = 7/15$.

(*b*) There are 4 numbers, 1, 2, 3, 4, which are less than 5, hence $p = 4/15$.

(*c*) There are 2 numbers, 2 and 4, which are even and less than 5; hence $p = 2/15$.

(*d*) By the addition rule (Theorem 3.6),

$$p = \frac{7}{15} + \frac{4}{15} - \frac{2}{15} = \frac{9}{15}$$

Alternately, there are 9 numbers, 1, 2, 3, 4, 6, 8, 10, 12, 14, which are even or less than 5; hence $p = 9/15$.

3.10. A box contains 2 white sox and 2 blue sox. Two sox are drawn at random. Find the probability p they are a match (same color).

There are $C(4,2) = \binom{4}{2} = 6$ ways to draw 2 of the sox. Only two pairs will yield a match. Thus $p = 2/6 = 1/3$.

3.11. Five horses are in a race. Audrey picks 2 of the horses at random and bets on them. Find the probability p that Audrey picked the winner.

There are $C(5,2) = \binom{5}{2} = 10$ ways to pick 2 of the horses. Four of the pairs will contain the winner. Thus, $p = 4/10 = 2/5$.

3.12. A class contains 10 men and 20 women of which half the men and half the women have brown eyes. Find the probability p that a person chosen at random is a man or has brown eyes.

Let $A = \{\text{men}\}$, $B = \{\text{brown eyes}\}$. We seek $P(A \cup B)$. First find:

$$P(A) = \frac{10}{30} = \frac{1}{3}, \quad P(B) = \frac{15}{30} = \frac{1}{2}, \quad P(A \cap B) = \frac{5}{30} = \frac{1}{6}$$

Thus, by the addition rule (Theorem 3.6),

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3}$$

3.13. Six married people are standing in a room. Two people are chosen at random. Find the probability p that: (*a*) they are married; (*b*) one is male and one is female.

There are $C(12,2) = 66$ ways to choose 2 people from the 12 people.

(*a*) There are 6 married couples; hence $p = 6/66 = 1/11$.

(*b*) There are 6 ways to choose the male and 6 ways to choose the female; hence $p = (6 \cdot 6)/66 = 36/66 = 6/11$.

3.14. Suppose 5 marbles are placed in 5 boxes at random. Find the probability p that exactly 1 of the boxes is empty.

There are exactly 5^5 ways to place the 5 marbles in the 5 boxes. If exactly 1 box is empty, then 1 box contains 2 marbles and each of the remaining boxes contains 1 marble. There are 5 ways to select the empty box, then 4 way to select the box containing 2 marbles, and $C(5,2) = 10$ ways to select 2 marbles to go into this box. Finally, there are 3! ways to distribute the remaining 3 marbles among the remaining 3 boxes. Thus

$$p = \frac{5 \cdot 4 \cdot 10 \cdot 3!}{5^5} = \frac{48}{125}$$

3.15. Two cards are drawn at random from an ordinary deck of 52 cards. (See Fig. 3-4.) Find the probability p that: (*a*) both are hearts, (*b*) one is a heart and one is a spade.

There are $C(52, 2) = 1326$ ways to choose 2 cards from the 52-card deck. In other words, $n(S) = 1326$.

(*a*) There are $C(13, 2) = 78$ ways to draw 2 hearts from the 13 hearts; hence

$$p = \frac{\text{number of ways 2 hearts can be drawn}}{\text{number of ways 2 cards can be drawn}} = \frac{78}{1326} = \frac{3}{51}$$

(*b*) There are 13 hearts and 13 spades, so there are $13 \cdot 13 = 169$ ways to draw a heart and a spade. Thus, $p = 169/1326 = 13/102$.

FINITE PROBABILITY SPACES

3.16. A sample space S consists of four elements, that is, $S = \{a_1, a_2, a_3, a_4\}$. Under which of the following functions P does S become a probability space?

(*a*) $P(a_1) = 0.4$, $P(a_2) = 0.3$, $P(a_3) = 0.2$, $P(a_4) = 0.3$.

(*b*) $P(a_1) = 0.4$, $P(a_2) = -0.2$, $P(a_3) = 0.7$, $P(a_4) = 0.1$.

(*c*) $P(a_1) = 0.4$, $P(a_2) = 0.2$, $P(a_3) = 0.1$, $P(a_4) = 0.3$.

(*d*) $P(a_1) = 0.4$, $P(a_2) = 0$, $P(a_3) = 0.5$, $P(a_4) = 0.1$.

(*a*) The sum of the values on the points in S exceeds one; hence P does not define S to be a probability space.

(*b*) Since $P(a_2)$ is negative, P does not define S to be a probability space.

(*c*) Each value is nonnegative and their sum is one; hence P does define S to be a probability space.

(*d*) Although $P(a_2) = 0$, each value is still nonnegative and their sum does equal. Thus, P does define S to be a probability space.

3.17. A coin is weighted so that heads is twice as likely to appear as tails. Find $P(T)$ and $P(H)$.

Let $P(T) = p$; then $P(\text{H}) = 2p$. Now set the sum of the probabilities equal to one, that is, set $p + 2p = 1$. Then $p = 1/3$. Thus $P(H) = 1/3$ and $P(B) = 2/3$.

3.18. Suppose A and B are events with $P(A) = 0.6$, $P(B) = 0.3$, and $P(A \cap B) = 0.2$. Find the probability that:

(*a*) A does not occur. (*c*) A or B occurs.

(*b*) B does not occur. (*d*) Neither A nor B occurs.

(*a*) By the complement rule, $P(\text{not } A) = P(A^c) = 1 - P(A) = 0.4$.

(*b*) By the complement rule, $P(\text{not } B) = P(B^c) = 1 - P(B) = 0.7$.

(*c*) By the addition rule,

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.6 + 0.3 - 0.2 = 0.7$$

(*d*) Recall [Fig. 3-7(*b*)] that neither A nor B is the complement of $A \cup B$. Therefore

$$P(\text{neither } A \text{ nor } B) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.7 = 0.3$$

3.19. A die is weighted so that the outcomes produce the following probability distribution:

Outcome	1	2	3	4	5	6
Probability	0.1	0.3	0.2	0.1	0.1	0.2

Consider the events:

$$A = \{\text{even number}\}, \quad B = \{2, 3, 4, 5\}, \quad C = \{x : x < 3\}, \quad D = \{x : x > 7\}$$

Find the following probabilities:

(*a*) $P(A)$ (*b*) $P(B)$ (*c*) $P(C)$ (*d*) $P(D)$

For any event E, find $P(E)$ by summing the probabilities of the elements in E.

(*a*) $A = \{2, 4, 6\}$, so $P(A) = 0.3 + 0.1 + 0.2 = 0.6$.
(*b*) $P(B) = 0.3 + 0.2 + 0.1 + 0.1 = 0.7$.
(*c*) $C = \{1, 2\}$, so $P(C) = 0.1 + 0.3 = 0.4$.
(*d*) $D = \varnothing$, the empty set. Hence $P(D) = 0$.

3.20. For the data in Problem 3.19, find: (*a*) $P(A \cap B)$, (*b*) $P(A \cup C)$, (*c*) $P(B \cap C)$.

First find the elements in the event, and then add the probabilities of the elements.

(*a*) $A \cap B = \{2, 4\}$, so $P(A \cap B) = 0.3 + 0.1 = 0.4$.
(*b*) $A \cup C = \{1, 2, 3, 4, 5\} = \{6\}^c$, so $P(A \cup C) = 1 - 0.2 = 0.8$.
(*c*) $B \cap C = \{2\}$, so $P(B \cap C) = 0.3$.

3.21. Let A and B be events such that $P(A \cup B) = 0.8$, $P(A) = 0.4$, and $P(A \cap B) = 0.3$. Find: (*a*) $P(A^c)$; (*b*) $P(B)$; (*c*) $P(A \cap B^c)$; (*d*) $P(A^c \cap B^c)$.

(*a*) By the complement rule, $P(A^c) = 1 - P(A) = 1 - 0.4 = 0.6$.
(*b*) By the addition rule, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Substitute in this formula to obtain:

$$0.8 = 0.4 + P(B) + 0.3 \quad \text{or} \quad P(B) = 0.1$$

(*c*) $P(A \cap B^c) = P(A \setminus B) = P(A) - P(A \cap B) = 0.4 - 0.3 = 0.1$.
(*d*) By DeMorgan's law, $(A \cup B)^c = A^c \cap B^c$. Thus

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.8 = 0.2$$

3.22. Suppose $S = \{a_1, a_2, a_3, a_4\}$, and suppose P is a probability function defined on S.

(*a*) Find $P(a_1)$ if $P(a_2) = 0.4$, $P(a_3) = 0.2$, $P(a_3) = 0.1$.
(*b*) Find $P(a_1)$ and $P(a_2)$ if $P(a_3) = P(a_4) = 0.2$ and $P(a_1) = 3P(a_2)$.

(*a*) Let $P(a_1) = p$. For P to be a probability function, the sum of the probabilities on the sample points must equal one. Thus, we have

$$p + 0.4 + 0.2 + 0.1 = 1 \quad \text{or} \quad p = 0.3$$

(*b*) Let $P(a_2) = p$ so $P(a_1) = 3p$. Thus

$$3p + p + 0.2 + 0.2 = 1 \quad \text{or} \quad p = 0.15$$

Hence $P(a_2) = 0.15$ and $P(a_1) = 0.45$.